On the Hilbert function of one-dimensional rings

Abstract

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The Hilbert function is one of the most classical objects associated to a graded ring in order to describe the dimension of the ring itself and the dimensions of its graded components. In this thesis we will focus on the Hilbert function of R when R is a one-dimensional local ring. More specifically, we consider a numerical semigroup ring, i.e. a subalgebra of $k[[t]]$ of the form $k[[t^{g_1}, \dots, t^{g_n}]]$ for some field k and coprime integers $g_1, \dots, g_n \in \mathbb{N}$. The numerical semigroup associated to R, which is the set of positive linear combinations of $g_1, \dots, g_n$, allows to translate and simplify algebraic and geometric properties of the tangent cone of the correspondent monomial curve in numerical terms.

In particular, the non-decrease of the Hilbert function can be expressed as the fact that there are more numbers that can be written in a maximal way as a sum of h elements chosen in $\{g_1, \dots, g_n\}$, than numbers that can be written in a maximal way as a sum of $h-1$ elements chosen in $\{g_1, \dots, g_n\}$. Although the simple formulation, there are many open problems for the non-decrease of the Hilbert function of a semigroup ring. We will move some steps towards the solution of some of them by considering the numerical invariants of the semigroup associated to the Apéry-set with respect to the multiplicity. We will be able to prove the non-decrease when the multiplicity is 8 and 9 (under one extra condition in the second case). More generally, we will give some necessary and sufficient numerical conditions for the non-decrease.

The original results presented in the thesis are contained in the paper *On the Hilbert function of the tangent cone of a monomial curve*, which is a joint work with Marco D'Anna and Vincenzo Micale, and will soon be published in the journal *Semigroup forum*.