Abstract of the thesis
"Turing patterns on multiplex networks"

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In 1952 Alan Turing published one of his last and most important works *"The chemical basis of morphogenesis"*. The article suggested that many phenomena that we observe in nature, from biological cellular differentiation to prey-predator model, can be described by the same reaction-diffusion equations system that involves two species, namely the *activators* and the *inhibitors*:

$$\begin{cases} \frac{d}{dt}u(t) & = & f(u(\boldsymbol{x},t),v(\boldsymbol{x},t))+D_{act}\nabla^2 u(\boldsymbol{x},t) \\ \frac{d}{dt}v(t) & = & g(u(\boldsymbol{x},t),v(\boldsymbol{x},t))+D_{inh}\nabla^2 v(\boldsymbol{x},t) \end{cases}$$

where $u(\boldsymbol{x},t)$ and $v(\boldsymbol{x},t)$ are the local densities in space $\boldsymbol{x}$ at time t for activators and inhibitors respectively and $D_{act}, D_{inh} > 0$ their corresponding diffusion constants.
Touring found that, under the condition $D_{act} < D_{inh}$, the reaction terms described by nonlinear functions f and g with the Laplacian diffusion operator ∇^2 can bring, after a small perturbation of the homogeneous equilibrium state, to a self-organized spatial rearrangement in patchy zones of activator-rich, inhibitor-poor, and in general to a spatial differentiation of the concentration u and v in the so-called Turing patterns.

In 2010 Nakao and Mikhailov in *Turing patterns in network-organized activator-inhibitor systems*, considered the situation in which the diffusion of the two species takes place over a complex network, and presented a linear stability analysis of the system around the uniform equilibrium point. In particular they found how the organization of the system into a Turing pattern, on a given network, is related to the ratio of the two diffusion constants.

Inspired by this work, we have generalized the linear stability analysis of reaction-diffusion systems to multiplex networks with two layers, where the two layers represent the two different substrates where the two species move. The results of our analysis provide a deeper understanding of the causes of instability in terms of the degree correlations between the two layers of the network, and in terms of the difference between the two diffusion constants. In particular, we have found that a multiplex networks having nodes with a high number of links in the inhibitors' layer and a low number of links in the activator layer is very susceptible to instabilities and consequently to form Turing patterns, independently from the value of the ratio of the diffusion constants, or from the global topology of the network. Our generalization of reaction-diffusion dynamics to multiplex networks has given us the possibility to show, for the first time, the formation of Turing patterns also in systems with $D_{inh} < D_{act}$. We have classified the initial perturbations according to the different Turing patterns formed, and for a subset of these we have also found a typical behaviour in the initial global dynamics that allows us to pass from an analysis *a priori* of the instability of the system, to a dynamical and statistical analysis of the system.

Finally, we have also considered the case in which the reaction dynamics can be node-depended. We have compared such heterogeneous case to the previous one, and we have seen that, although the causes of the instability remain essentially the same, the effects can be very different. In particular, we have noted that the stable patterns obtained have strong positive correlations among them, while in the previous homogeneous case we had, in most of the cases, only two possible patterns, one being the opposite of the other.